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Chaotic diffusion in the action and frequency domains: estimate of instability times

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Abstract. In this work, we explore the problem of chaotic diffusion in dynamical systems within two different frameworks: the action and the frequency domains. Dynamical characterization of the phase space of the Planar, Circular, Restricted Three Body Problem (PCR3BP) is made using traditional tools (Poincaré Sections, Lyapunov Exponents and Spectral Analysis). The results obtained are compared to those obtained applying the Shannon Entropy method (SE), Mean Squared Displacement (MSD) of independent frequencies and Laskar's Equation of Diffusion. The diffusion/Instability times present both qualitative and quantitative agreement with instability times obtained through direct integrations. We conclude that the study of instabilities in the frequency domain provides reliable estimates of the diffusion timescales, and also presents a good cost-benefit in terms of computation-time.

Resumo. Neste trabalho, exploramos o problema da difusão caótica em sistemas dinâmicos por dois referenciais diferentes: o espaço de ações e o espaço de frequências. O espaço de fase do Problema Planar, Circular e Restrito de Três Corpos (PCR3BP) é feita com o uso de ferramentas tradicionais (Seções de Poincaré, Expoentes de Lyapunov e Análise Espectral) e comparada com a obtida pelos Coeficientes de Difusão calculados por meio da Entropia de Shannon (SE), Deslocamento Quadrático Médio da evolução de frequências independentes (MSD) e da Equação de Difusão de Laskar aplicada no espaço de frequências. Os tempos de difusão/instabilidade calculados a partir desses coeficientes apresentam concordância qualitativa e quantitativa com os tempos de instabilidade obtidos por integrações diretas, mas concluímos que o estudo de instabilidades no domínio da frequência fornece estimativas confiáveis para escalas de tempo de difusão, além de apresentar um bom custo-benefício em termos de tempo de computação e confiabilidade.

Keywords. Chaos – Celestial Mechanics

1. Introduction

Chaotic diffusion in the action space is a commonly used approach to study stability of dynamical systems (Froeschlé et al., 2005; Cachucho et al., 2010; Martí et al., 2016). However, the analysis of the diffusion in the frequency domain shows some advantages; indeed, Laskar (1990) showed that the existence of a chaotic zone is much more visible in the frequency domain than in the action domain. In addition, it is also independent on the choice of coordinates.

Here, we investigate chaotic diffusion applying the different tools within both domains, estimating chaotic diffusion coefficients and diffusion timescales within regions of the phase space of the PCR3BP. Finally, we compare the calculated timescales with instability times obtained through the calculation of the Lyapunov times and direct integrations of equations of motion.

2. Methods and Procedures

We perform a dynamical characterizarion of the phase space of the PCR3BP using several methods, such as the Poincaré Surface of Section, the Fast Lyapunov Indicator (FLI, Froeschlé et al., 2005) and the Method of the Spectral Number (SN, Michtchenko & Ferraz-Mello, 2001).

Together with the traditional methods, we calculate diffusion coefficients using the tools, which represent both frequency and action domains: The Spectral Analysis (SA), the Wavelet Analysis (WA) and the Shannon Entropy (SE) methods.

Finally, estimates of Diffusion Timescales defined by the diffusion coefficients are compared to those obtained through direct integrations and the calculation of the Lyapunov times.

3. Diffusion Coefficients and Instability Times

Concerning the behavior of the system in the frequency domain, Laskar (1993) proposed that diffusive phenomena tend to follow traditional Equation of Diffusion $\partial_{xx} f(x,t) \propto \partial_t f(x,t)$: the diffusion in time can be related to the diffusion in space.

Cincotta & Simó (2000) stated that the diffusion coefficients in the frequency space should follow Chirikov (1979) theory of diffusion and can be obtained by means of the MSD. Here we adopt the definition of the diffusion coefficients introduced by Marzari et al. (2003) and introduce one based on the work of Froeschlé (2005).

Working in the action domain, we use the definition given in Beaugé & Cincotta (2019) and Alves et al. (2021), which estimates the diffusion coefficients D_S based on the time evolution of the Shannon Entropy.

3.1. Diffusion in the Phase Space

In this work, 201 initial conditions were integrated over 10.000 orbital periods in the interval $0.05 \leq x(0) \leq 0.75$, with fixed $\dot{x}(0) = 0$, $y(0) = 0$ and $\dot{y}(0)$ calculated for $C_J = 3.03$ and $\mu = 0.0009537$. For those 201 particles, the FLI, the SN and the time evolution of the independent frequencies were calculated applying the WAM. An ensemble of the 5 particles, close to the initial condition, was also integrated over 1000 orbital periods, in order to obtain the SE. For all particles from the ensemble, the complete Spectral Analysis of the orbital motion was also done.

The main results obtained are shown in in Figure 1.

Several dynamical structures can be seem from the application of two robust chaos indicators, FLI and SN, specially nearer

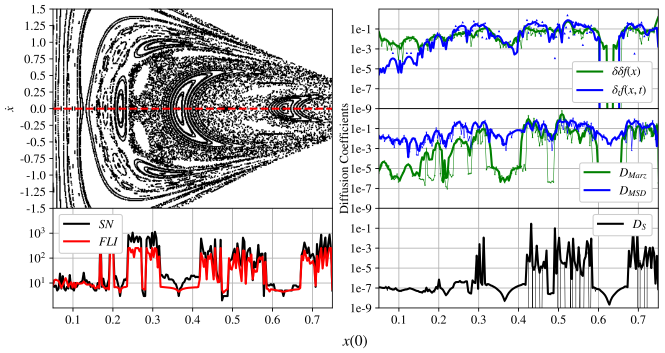

Figure 1. Top left: Poincaré sections, Top right: Diffusion Coefficients obtained using Laskar's Equation of Diffusion, Middle right: Diffusion coefficients obtained using the MSD, Bottom right: Diffusion Coefficients obtained using the SE, Bottom left: Space evolution of the FLI and the SN.

m_1 (bottom-left panel), where mainly regular motions can be observed on the Poincaré section.

The diffusion coefficients obtained using the Laskar's relation (top-right panel) show the expected behaviour inside a region of confined chaos, despite a visible spread of the $\delta_t f(x,t)$ solutions.

The coefficients based on the time evolution of frequencies show similar behavior (middle-right panel). Nonetheless, the Marzari's method shows sensibility in the slow-diffusion regions of the phase space, when compared to the Froeschlé method.

The SE allows us to clearly distinguish between regions of the regular and chaotic motion, being more sensitive inside the regions, where slow diffusion dominates (bottom-right panel).

3.2. Estimate of Instability Times

Using the values of the diffusion coefficients obtained above, we can estimate the corresponding diffusion times (Froeschlé et al., 2005; Robutel & Gabern, 2006; Alves et al., 2021). The values obtained are shown in Figure 2 and compared to the results obtained through the direct integrations (solid black and red lines on the top-left panel).

In the region close to m_1, the instability times obtained through the numerical integrations do not corroborate the results provided by the Chaos indicators, once chaotic motion is confined and slowly diffusing, as shown on the previous session. Farther from m_1, the Lyapunov times are related to the escape/crossing times, following results obtained by Lecar et al. (1992), specially in regions of strong chaotic motion.

The instability times obtained using the Laskar's equation of diffusion (top-right panel) mostly agree with the direct integration times and also show robustness in relation to the use of either a single particle or ensembles (black and red/blue curves, respectively), obtained by the SA or the WA (black/red and rlue curves, respectively).

The coefficients obtained using the MSD (bottom-left panel) show agreement with the direct integration times, Marzari's coefficient more coincident than those given by the Froeschlé's definition.

The instability estimates obtained by the use of the SE (bottom-right panel) agree with those obtained by the MSD method, but show higher dispersion than the others in the regions of the strong chaotic motion, mostly due to the choice of the rescaling constant defined in Beaugé and Cincotta (2019) and Alves et al. (2021).

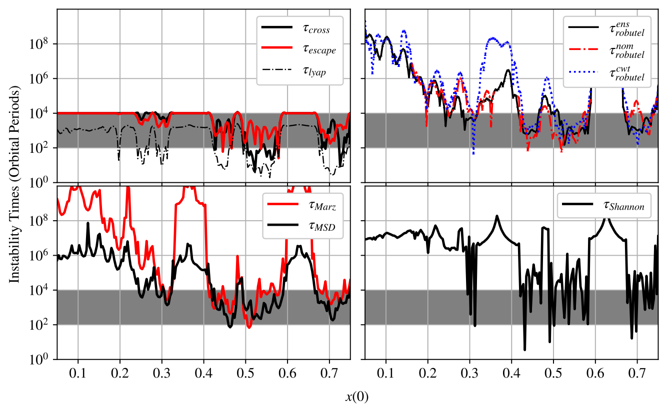

Figure 2. Instability times obtained by distinct methods. Top left: Direct integrations (red curve for escape/collision times, black solid curve for orbit-crossing times and black dashed for Lyapunov times). Top right: Using Robutel & Gabern (2006) approach, using an ensemble of particles (solid black curve), one particle (dashed red curve) and mean frequencies using WAM (dashed blue curve). Bottom left: Using Marzari's Method (τ_{marz}) and Froeschlé's (τ_{MSD}). Bottom right: Using coefficients obtained by SE. Gray region show typical instability times obtained by direct integrations.

4. Conclusions

The use of the different tools for the characterization of dynamical systems ensures greater accuracy, however, the economy of computational resources should be kept in mind during the choice of a tool to be applied.

The study of chaotic diffusion in the frequency space is shown to be of good cost-benefit, in terms of both accuracy and computation times. On the contrary, in our case, the application of the SE proved be computationally time-consuming.

Anyway, it seems that the results yielded by the use of different tools show a good agreement with the instability times obtained through the numerical integrations.

Further studies must be done in order to verify robustness of the methods, as well as the longer integration times and additional applications to the different dynamical systems; nonetheless, our results indicate the reliability of the tested tools for a fast and reliable dynamical characterization of system.

Further development of the presented tools should also play an important role, in order to increase their accuracy and cost-benefits.

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