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Rotating spacetimes

Kerr black holes, Teo wormholes and Bobrick warp drive

M. Raia Neto[1] & L. Claudio Lima Botti[2]

[1] Universidade Federal de São Carlos e-mail: mrneto@estudante.ufscar.br
[2] Instituto Nacional de Pesquisas Espaciais e-mail: luiz.botti@inpe.br

Abstract. We studied important theoretical facts about black holes such as the frame-dragging phenomena and the basic structure of Kerr black holes, and we discussed that rotation increases the violation of energy conditions on wormholes and warp drives rotating.

Resumo. No presente artigo, foram estudados fatos teóricos importantes sobre os buracos negros, como o fenômeno do arrasto de referênciais e a estrutura básica dos buracos negros de Kerr, e discutimos também que a rotação aumenta a violação das condições de energia em buracos de minhoca e dobas espaciais.

Keywords. Black hole physics – Gravitation – Relativistic processes

1. Introduction

Gravity is an essential interaction to study astrophysical phenomena, and it is a fact that astrophysical bodies do exhibit rotation. In particular, when we study relativistic effects near compact objects, such as black holes and neutron stars, the geometry of spacetime[1] can be described by a general axisymmetric metric tensor (Islam 1985):

$$ds^2 = g_{00}(dx^0)^2 + 2g_{03}dx^0dx^3 + g_{33}(dx^3)^2 + g_{ij}dx^idx^j, \qquad (1)$$

where i, j are to be summed over values of $1, 2$: the other spatial terms of the metric. As the one can visualize in (1), the mere presence of the rotation of a body affects the spacetime geometry. In the present paper we studied three soluitons: Kerr black holes (Kerr 1963), Teo wormholes (Teo 1998) and Bobrick warp drives (Bobrick & Martire 2021). All of them are, respectively, rotating versions of Schwarzschild black hole, Morris-Thorne wormhole (Thorne & Morris 1988) and Alcubierre warp drive (Alcubierre 1994). As we shall see, the mere presence of rotation affects the structure of black holes and the energy conditions.

2. Kerr black holes

Kerr black holes are rotating exterior solutions of a black hole with angular momentum J. This spacetime was discovered by Kerr (1963) and have the following metric, in Boyer-Lindquist coodinates:

$$ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)dt^2+\frac{\Sigma}{\Delta}dr^2+\Sigma d\theta^2+\left(\frac{A\sin^2\theta}{\Sigma}\right)d\phi^2-g_{t\phi}dtd\phi. \qquad (2)$$

Where the metric potentials are given by $g_{t\phi} = \frac{4Mra\sin^2\theta}{\Sigma}$, $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. It is important to realize that the black hole rotates about the polar axis and the angular momentum of the black hole are encoded in the geometry of spacetime via the parameter $a = \frac{J}{M}$. The first feature of this spacetime appears when we impose the limit $J \to 0 \iff a \to 0$; we recover the spherical symmetry of the body. Knowing that black holes are rotating objects in nature and the fact about the previous limit on angular momentum, it is possible to say then that Schwarzschild geometry is just a approximation for the exterior spacetime of real black holes. The event horizons are given by the coordinate singularity

$$\Delta = 0 \iff r = r_\pm = M \pm \sqrt{M^2 - a^2}, \qquad (3)$$

and the ring shaped singularity at

$$\Sigma = 0 \iff r = 0 \text{ and } \cos\theta = 0. \qquad (4)$$

The topology of the singularity of Kerr black holes are drastically different from Schwarzschild ones; the spherically symmetric black holes exhibit a point-like singularity at $r = 0$ and just one event horizon at $r = 2M$. An important fact to note about the singularities of rotating black holes, is that they are avoidable. Mathematically, because of the condition (4), it is direct to see that the singularity occurs just when the angle $\theta = \frac{\pi}{2}$; the interpretation for this is that for any other angle an observer who falls into the Kerr black hole will avoid, in principle, the singularity.

Now, perhaps the most important physical region that appears in a rotating black hole lies outside the event horizons: the ergoregion. In this region, it is impossible for a test particle to remain stationary with respect to observers at infinity. The reason for that is since the black hole is rotating, this ergoregion appears and induces a *frame-dragging* effect. Since the Kerr spacetime have two killing vectors $K^0 = (1, 0, 0, 0)$ and $K^3 = (0, 0, 0, 1)$, and constants of motion for the energy E and angular momentum L_z: $E = g_{00}K^0u^0 = g_{00}u^0$ and $-L_z = g_{33}K^3u^3 = g_{33}u^3$,

it is possible, to use the constants of motion, and analyse the motion of a observer with zero angular momentum, falling from infinity $L_z = 0$. Therefore, que equations of motion are:

$$\frac{dt}{d\tau} = \frac{(r^2 - a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma\Delta} \text{ and } \frac{d\phi}{d\tau} = \frac{2MarE}{\Sigma\Delta}. \qquad (5)$$

The frame-dragging effect arises when we realize that the angular velocity seen by a distant observer is:

$$\frac{d\phi}{dt} = \frac{\dot\phi}{\dot t} = \frac{2Mar}{(r^2 - a^2)^2 - a^2\Delta\sin^2\theta}. \qquad (6)$$

[1] The metric convention is given by: $(-, +, +, +)$.

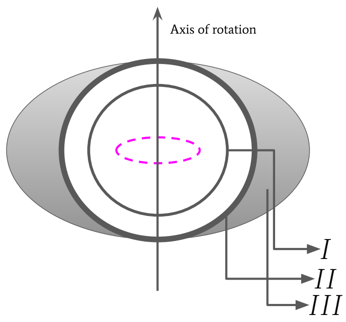

Figure 1. *I* is the inner event horizon r_-, *II* is the outer event horizon r_+ and *III* is ergoregion.

So, the observer gained angular velocity when she reached the vicinity of the black hole.

3. Energy Conditions

It is quite direct to define the energy-momentum tensor via the Einstein curvature of spacetime, simply as $T_{\mu\nu} := (1/8\pi)G_{\mu\nu}$. This arbitrariness is solved when we impose constrains on which type of matter field is physically reasonable (Morris & Thorne 1988).These constrains are called energy conditions. There are four of them (Visser 1994) but, in the present study, just the null energy condition and weak energy condition will be important. We have the following energy conditions (Visser 1994): 1) for the light-like vectors ℓ^α, we have the null energy condition: $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$, 2) for every time-like vector t^α the condition upon classical matter should satisfy the weak energy condition: $T_{\mu\nu}t^\mu t^\nu \geq 0$.

4. Teo Wormholes

A traversable wormhole with rotation is given by a modified Morris-Thorne wormhole proposed by Teo (1998):

$$ds^2 = -N^2dt^2 + \left(1 - \frac{b}{r}\right)^{-1} dr^2 + r^2K^2(d\theta^2 + \sin^2\theta(d\phi - \omega dt)^2), \quad (7)$$

where the functions N, K, b are functions of r and θ (Teo 1998). This is an axisymmetric spacetime which connects two assymptotic regions of the manifold. The matter that violates the energy conditions is supposed to be close to the throat of the wormhole (Morris & Thorne 1988). Therefore, we shall compare, with equations (9) and (10), the violation of null energy condition for rotating and non-rotating wormholes for a shape function given by $b(r) = r_0^2/r$, near the throat, $b(r)|_{r_0} = r_0$. Also, we considered a simplified version of the rotating wormhole, where the metric coeficients are given by, $N = K = 1$. The throat have occurs at the value of $r_0 = 1$. For the non-rotating wormhole the null energy condition is given by:

$$T_{\mu\nu}\ell^\mu\ell^\nu = -\frac{1}{8\pi r_0^2} \approx -0,04. \quad (8)$$

But, for the rotating case the null energy conditon is almost the same:

$$T_{\mu\nu}\ell^\mu\ell^\nu = -\frac{1}{4\pi r_0^2} \approx -0,08. \quad (9)$$

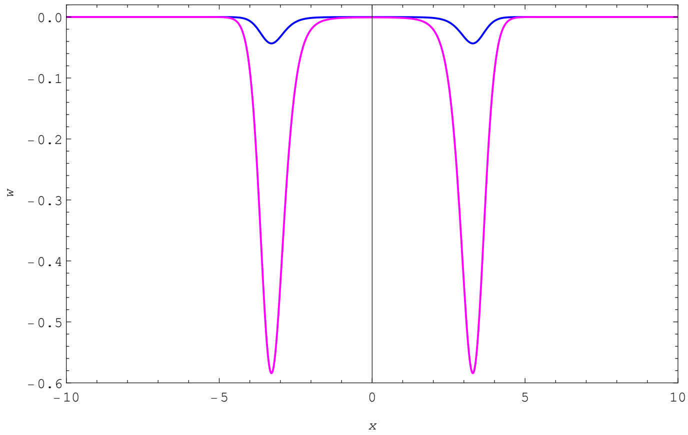

Figure 2. Comparisson between Alcubierre and Bobrick warp drives. The blue curve is the energy density for the non-rotating warp drive and the magenta curve is for rotating case.

5. Bobrick warp drive

Warp drives are solutions of Einstein field equations that allows superluminal travel powered by spacetime curvature (Bobrick & Martire 2021). The first warp drive as proposed by Alcubierre (1994) and it was a non-rotating metric:

$$ds^2 = -dt^2 + (dx - v_s f dt)^2 + d\rho^2 + (\rho d\theta - f\rho\omega_s dt)^2. \quad (10)$$

The energy density was given by (Bobrick & Martire 2021), for the non-rotating case:

$$T_{\mu\nu}t^\mu t^\nu = -\frac{1}{8\pi}\frac{v_2^2}{4}\left(\frac{\partial f}{\partial\rho}\right)^2, \quad (11)$$

and for the rotating case:

$$T_{\mu\nu}t^\mu t^\nu = -\frac{1}{8\pi}\left(\frac{v_2^2}{4}\left(\partial_\rho f\right)^2 + \frac{\rho^2\omega_s^2}{4}\left((\partial_\rho f)^2 + (\partial_x f)^2\right)\right). \quad (12)$$

Where, f is the Alcubierre top-hat function, ω_s is the angular velocity of the warp drive and v_s is the linear velocity. The curves on Figure 2, are the plots of the energy densities. The rotation of spacetime therefore increases the violation of weak energy condition. This have an interpretation that, spinning warp drives require more "exotic matter" to exist.

6. Conclusions

The presence of rotation induces dramatic changes in black holes: they have more than just one event horizon and a singularity with different topology. Also, the analysis of energy conditions of the rotating wormhole and warp drive, revealed that the rotation increases the matter violation near the rotating wormhole throat. The same behavior occurred with the rotating warp drive, the rotation induces an increase of the negative energy density.

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